CHAPTER 4: LINEAR PROGRAMMING: MODELING EXAMPLES

Overview

In this chapter, more complex examples of model formulation are presented. These examples have been selected to illustrate some of the more popular application areas of linear programming. They also provide guidelines for model formulation for a variety of problems and computer solutions with Excel and QM for Windows. You will notice as you go through each example that the model formulation is presented in a systematic format. First, decision variables are identified, then the objective function is formulated, and finally the model constraints are developed. Model formulation can be difficult and complicated, and it is usually beneficial to follow this set of steps in which you identify a specific model component at each step instead of trying to "see" the whole formulation after the first reading.

Learning Outcomes

At the end of this lesson, students must be able to:

- ✓ Apply model formulation in different application areas of linear programming

COURSE MATERIALS

A Product Mix Example

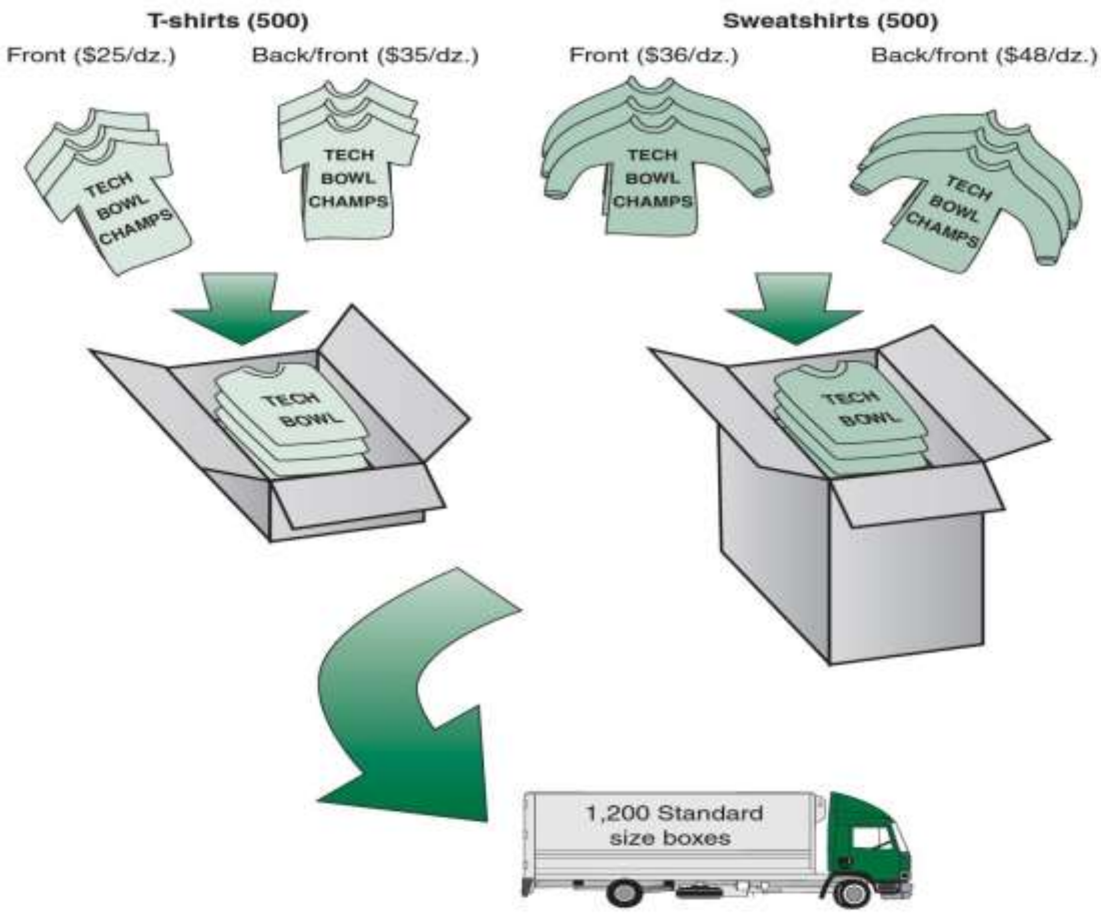
Quick-Screen is a clothing manufacturing company that specializes in producing commemorative shirts immediately following major sporting events such as the World Series, Super Bowl, and Final Four. The company has been contracted to produce a standard set of shirts for the winning team, either State University or Tech, following a college football bowl game on New Year's Day.

tThe company has to complete all production within 72 hours after the game, at which time a trailer truck will pick up the shirts. The company will work around the clock. The truck has enough capacity to accommodate 1,200 standard-size boxes. A standardsize box holds 12 T-shirts, and a box of 12 sweatshirts is three times the size of a standard box. The company has budgeted $25,000 for the production run. It has 500 dozen blank sweatshirts and T-shirts each in stock, ready for production. This scenario is illustrated in Figure 4.1.

	Processing Time (hr.) per Dozen	Cost per Dozen	Profit per Dozen
Sweatshirt—F	0.10	$36	$90
Sweatshirt—B/F	0.25	48	125
T-shirt—F	0.08	25	45
T-shirt—B/F	0.21	35	65

The company wants to know how many dozen (boxes) of each type of shirt to produce in order to maximize profit.

Figure 4.1 Quick-Screen Shirts



Summary of Linear Programming Model Formulation Steps

Step 1: Define the decision variables

How many (dozens of) T-shirts and sweatshirts of each type to produce

Step 2: Define the objective function

Maximize profit

Step 3: Define the constraints

The resources available, including processing time, blank shirts, budget, and shipping capacity

Decision Variables

This problem contains four decision variables, representing the number of dozens (boxes) of each type of shirt to produce:

x_1 = sweatshirts, front printing

x_2 = sweatshirts, back and front printing

x_3 = T-shirts, front printing

x_4 = T-shirts, back and front printing

The Objective Function

The company's objective is to maximize profit. The total profit is the sum of the individual profits gained from each type of shirt. The objective function is expressed as

$$\text{maximize } Z = \$90x1 + 125x2 + 45x3 + 65x4$$

Model Constraints

The first constraint is for processing time. The total available processing time is the 72-hour period between the end of the game and the truck pickup:

$$0.10x + 0.25x_2 + 0.08x_3 + 0.21x_4 \leq 72 \text{ hr}$$

The second constraint is for the available shipping capacity, which is 1,200 standard-size boxes. A box of sweatshirts is three times the size of a standard-size box. Thus, each box of sweatshirts is equivalent in size to three boxes of T-shirts. This relative size differential is expressed in the following constraint:

$3x1 + 3x2 + x3 + x4 \leq 1{,}200$ boxes

The third constraint is for the cost budget. The total budget available for production is $25,000:

$\$36x1 + 48x2 + 25x3 + 35x4 \leq \$25{,}000$

The last two constraints reflect the available blank sweatshirts and T-shirts the company has in storage:

$x1 + x2 \leq 500$ dozen sweatshirts

$x3 + x4 \leq 500$ dozen T-shirts

Model Summary

The linear programming model for Quick-Screen is summarized as follows:

maximize $Z = 90x1 + 125x2 + 45x3 + 65x4$

subject to

$0.10x_1 + 0.25x_2 + 0.08x_3 + 0.21x_4 \leq 72$

$3x_1 + 3x_2 + x_3 + x_4 \leq 1,200$

$36x_1 + 48x_2 + 25x_3 + 35x_4 \leq 25,000$

$x_1 + x_2 \leq 500$

$x_3 + x_4 \leq 500$

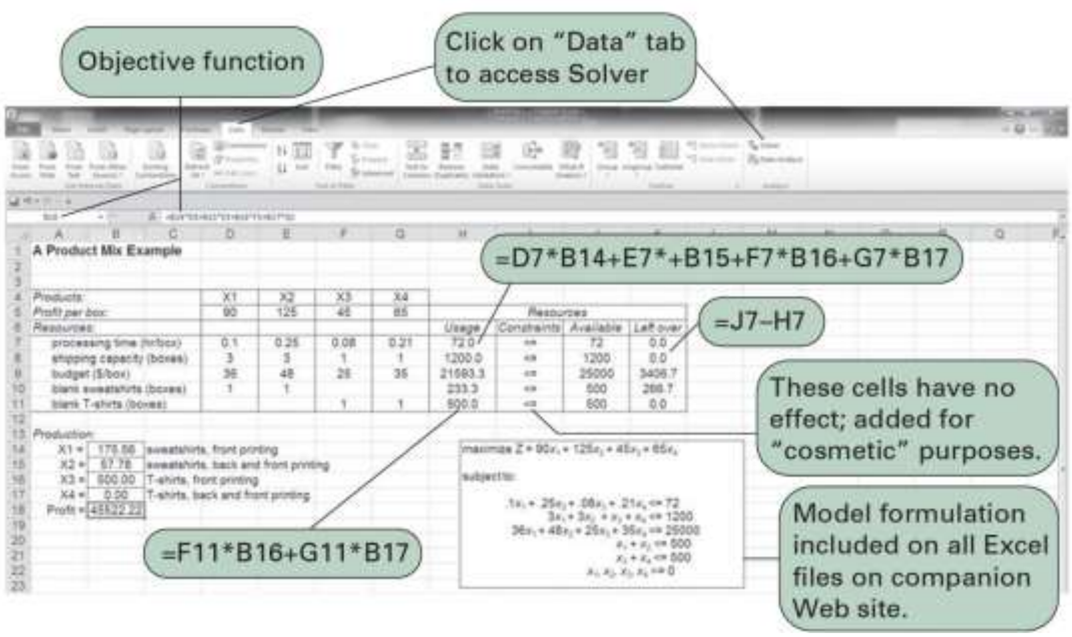
$x_1, x_2, x_3, x_4 \geq 0$

Computer Solution with Excel

The Excel spreadsheet solution for this product mix example is shown in Exhibit 4.1. The decision variables are located in cells B14:B17. The profit is computed in cell B18, and the formula for profit, **=B14*D5+B15*E5+B16*F5+B17*G5**, is shown on the formula bar at the top of the spreadsheet. The constraint formulas are embedded in cells H7 through H11, under the column titled "Usage." For example, the constraint formula for processing time in cell H7 **is =D7*B14+E7*B15+F7*B16+G7*B17**. Cells H8 through H11 have similar formulas.

Cells K7 through K11 contain the formulas for the leftover resources, or slack. For example, cell K7 contains the formula **=J7–H7**. These formulas for leftover resources enable us to

Exhibit 4.1



demonstrate a spreadsheet operation that can save you time in developing the spreadsheet model. First, enter the formula for leftover resources, J7–H7, in cell K7, as we have already shown. Next, using the right mouse button, click on "Copy." Then cover cells K8:K11 with the cursor (by holding the left mouse button down). Click the right mouse button again and then click on "Paste." This will automatically insert the correct formulas for leftover resources in cells K8 through K11 so that you do not have to type them all in individually. This copying operation can be used when the variables in the formula are all in the same row or column. The copying operation simply increases the row number for each cell that the formulas are copied into (i.e., J8 and H8, J9 and H9, J10 and H10, and J11 and H11).

Also, note the model formulation in the box in the lower right-hand corner of the spreadsheet in Exhibit 4.1. The model formulations for the remaining linear programming models in this and other chapters are included on the Excel files on the companion website accompanying this text.

The Solver Parameters window for this model is shown in Exhibit 4.2. Notice that we were able to insert all five constraint formulas with one line in the “Subject to the Constraints:” box. We used the constraint H7:H11 <= J7:J11, which means that all the constraint usage values computed in cells H7 through H11 are less than the corresponding available resource values computed in cells J7 through J11.

Computer Solution with QM for Windows

The QM for Windows solution for this problem is shown in Exhibit 4.3.

Solution Analysis

The model solution is

x_1 = 175.56 boxes front-only sweatshirts sweatshirts.

x_2 = 57.78 boxes of front and back sweatshirts

x_3 = 500 boxes of front-only T-shirts

Z = \$45,522.22 profit

The manager of Quick-Screen might have to round off the solution to send whole boxes—for example, 175 boxes of front-only sweatshirts, 57 of front and back sweatshirts, and 500 of front-only

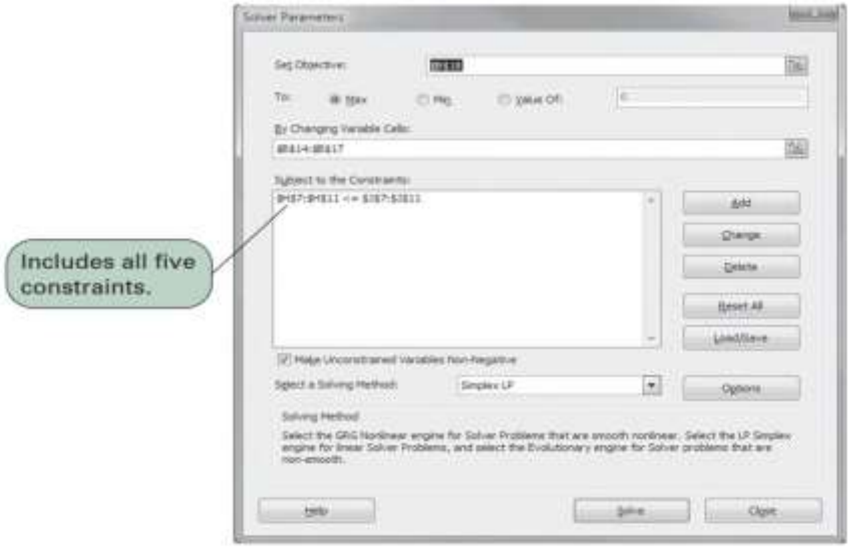


Exhibit 4.2

Exhibit 4.3

Original Problem w/answers

Product Mix Example Solution

	X1	X2	X3	X4		RHS	Dual
Maximize	90	125	45	65			
Processing time (hrs)	.1	.25	.08	.21	<=	72	233.3333
Shipping capacity (boxes)	3	3	1	1	<=	1,200	22.2222
Budget ($)	36	48	25	35	<=	25,000	0
Blank sweats (dozens)	1	1	0	0	<=	500	0
Blank T's (dozens)	0	0	1	1	<=	500	4.1111
Solution->	175.5556	57.7778	500	0	Optimal Z->	45,522.22	

T-shirts. This would result in a profit of $45,375.00, which is only $147.22 less than the optimal profit value of $45,522.22.

Sensitivity Analysis

After formulating and solving this model, Quick-Screen might decide that it needs to produce and ship at least some of each type of shirt. Management could evaluate this possibility by adding four constraints that establish minimum levels of production for each type of shirt, including front and back T-shirts, x_4, none of which are produced in the current solution. The manager might also like to experiment with the constraints to see the effect on the solution of adding resources. For example, looking at the Ranging window for QM for windows in Exhibit 4.4, the dual value for processing time shows profit would increase by $233.33 per hour (up to 98.33 hours, the upper limit

Ranging

Product Mix Example Solution

Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X1	175.5556	0	90	50	101.9231
X2	57.7778	0	125	113.0769	138.2143
X3	500	0	45	40.8889	Infinity
X4	0	10.3333	65	-Infinity	75.3333
Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Processing time (hrs)	233.3333	0	72	63.3333	98.3333
Shipping capacity (boxes)	22.2222	0	1,200	884	1,460
Budget ($)	0	3,406.666	25,000	21,593.33	Infinity
Blank sweats (dozens)	0	266.6667	500	233.3333	Infinity
Blank T's (dozens)	4.1111	0	500	0	685.7144

of the sensitivity range for this constraint quality value). Although the 72-hour limit seems pretty strict, it might be possible to reduce individual processing times and achieve the same result.

A Marketing Example

The Biggs Department Store chain has hired an advertising firm to determine the types and amount of advertising it should invest in for its stores. The three types of advertising

	Exposure (people/ad or commercial)	Cost
Television commercial	20,000	$15,000
Radio commercial	12,000	6,000
Newspaper ad	9,000	4,000

availableare television and radio commercials and newspaper ads. The retail chain desires to know thenumber of each type of advertisement it should purchase in order to maximize exposure. It is estimated that each ad or commercial will reach the following potential audience and cost the following amount:

The company must consider the following resource constraints:

1. The budget limit for advertising is $100,000.

2. The television station has time available for 4 commercials.

3. The radio station has time available for 10 commercials.

4. The newspaper has space available for 7 ads.

5. The advertising agency has time and staff available for producing no more than a total of 15 commercials and/or ads.

Decision Variables

This model consists of three decision variables that represent the number of each type of advertising produced:

x_1 = number of television commercials

x_2 = number of radio commercials

x_3 = number of newspaper ads

The Objective Function

The objective of this problem is different from the objectives in the previous examples, in which only profit was to be maximized (or cost minimized). In this problem, profit is not to be maximized; instead, audience exposure is to be maximized. Thus, this objective function demonstrates that although a linear programming model must either maximize or minimize some objective, the objective itself can be in terms of any type of activity or valuation. For this problem the objective audience exposure is determined by summing the audience exposure gained from each type of advertising:

$$\text{maximize } Z = 20{,}000x_1 + 12{,}000x_2 + 9{,}000x_3$$

where

Z = total level of audience exposure

$20{,}000x_1$ = estimated number of people reached by television commercials

$12{,}000x_2$ = estimated number of people reached by radio commercials

$9{,}000x_3$ = estimated number of people reached by newspaper ads

Model Constraints

The first constraint in this model reflects the limited budget of $100,000 allocated for advertisement:

$$\$15{,}000x_1 + 6{,}000x_2 + 4{,}000x_3 \leq \$100{,}000$$

where

$\$15{,}000x_1$ = amount spent for television advertising

$6{,}000x_2$ = amount spent for radio advertising

$4{,}000x_3$ = amount spent for newspaper advertising

The next three constraints represent the fact that television and radio commercials are limited to 4 and 10, respectively, and newspaper ads are limited to 7:

$x_1 \leq 4$ television commercials

$x_2 \leq 10$ radio commercials

$x_3 \leq 7$ newspaper ads

The final constraint specifies that the total number of commercials and ads cannot exceed 15 because of the limitations of the advertising firm:

$x_1 + x_2 + x_3 \leq 15$ commercials and ads

Model Summary

The complete linear programming model for this problem is summarized as

maximize $Z = 20{,}000x_1 + 12{,}000x_2 + 9{,}000x_3$

subject to

$\$15{,}000x_1 + 6{,}000x_2 + 4{,}000x_3 \leq \$100{,}000$

$x_1 \leq 4$

$x_2 \leq 10$

$x_3 \leq 7$

$x_1 + x_2 + x_3 \leq 15$

$x_1, x_2, x_3 \geq 0$

Computer Solution with Excel

The solution to our marketing example using Excel is shown in Exhibit 4.5. The model decision variables are contained in cells D6:D8. The formula for the objective function in cell E10 is shown on the formula bar at the top of the screen. When Solver is accessed, as shown in Exhibit 4.6, it is necessary to use only one formula to enter the model constraints: H6:H10 <= J6:J10.

Solution Analysis

The solution shows

$x_1 = 1.82$ television commercials

$x_2 = 10$ radio commercials

$x_3 = 3.18$ newspaper ads

Z = 185,000 audience exposure

This is a case where a non-integer solution can create difficulties. It is not realistic to round 1.82 television commercials to 2 television commercials, with 10 radio commercials and 3 newspaper ads. Some quick arithmetic with the budget constraint shows that such a solution will exceed the

Exhibit 4.5

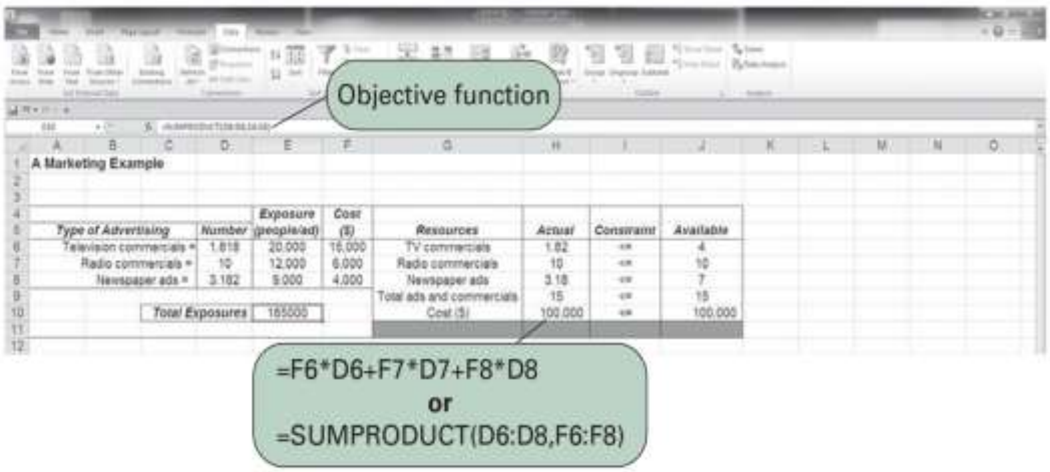


Exhibit 4.6

$100,000 budget limitation, although by only $2,000. Thus, the store must either increase its advertising or plan for 1 television commercial, 10 radio commercials, and 3 newspaper ads. The audience exposure for this solution will be 167,000 people, or 18,000 fewer than the optimal number, almost a 10% decrease. There may, in fact, be a better solution than this “rounded-down” solution.

Exhibit 4.7

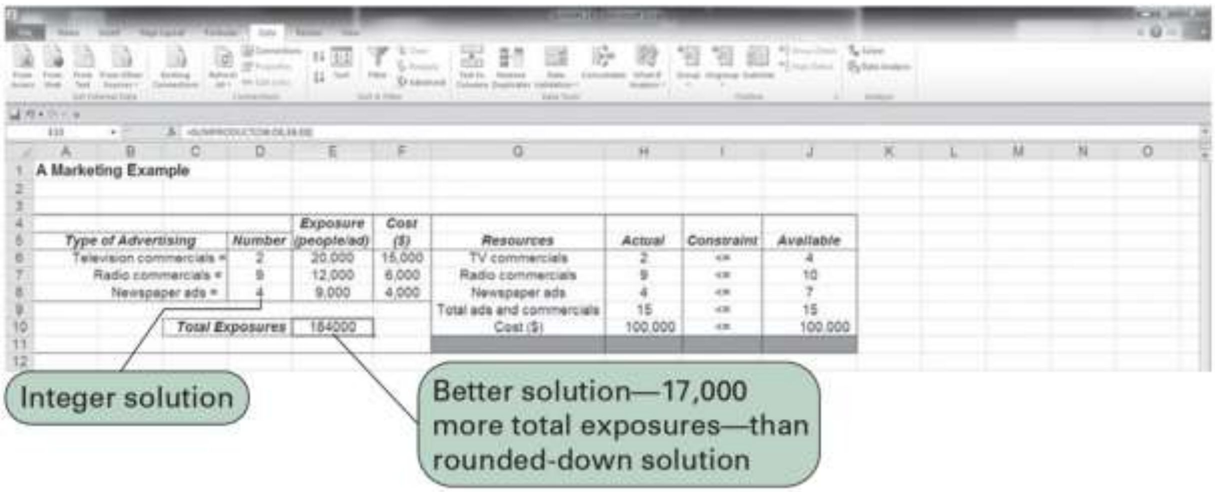
Exhibit 4.8

The integer linear programming technique, which restricts solutions to integer values, should be used. Although we will discuss the topic of integer programming in more detail in Chapter 5, for now we can derive an integer solution by using Excel with a simple change when we input our constraints in the Solver Parameters window. We specify that our decision variable cells, **D6:D8**, are integers in the Change Constraint window, as shown in Exhibits 4.7 and 4.8. This will result in the spreadsheet solution in Exhibit 4.9 when the problem is solved, which you will notice is better (i.e., 17,000 more total exposures) than the rounded-down solution.

Generally, it is not feasible to attempt to “see” the whole formulation of the constraints and objective function at once, following the definition of the decision variables. A more prudent approach is to construct the objective function first (without direct concern for the constraints) and then to direct attention to each problem restriction and its corresponding model constraint. This is a systematic approach to model formulation, in which steps are taken one at a time.

Formulating a linear programming model from a written problem statement is often difficult. The steps for model formulation described in this section are generally followed; however, the problem must first be defined (i.e., a problem statement or some similar descriptive apparatus must be developed). Developing such a statement can be a formidable task, requiring the assistance of many individuals and units within an organization.

Exhibit 4.9



Activities/Assesments

1. Apply Linear Programming modelling technique in production wherein the firm has to make the proper combination of two technologies to produce product Y. the mix should enable the company to product this product at the lowest cost possible. Provide your own ficitous data
2. On their farm, the Friendly family grows apples that they harvest each fall and make into three products—apple butter, applesauce, and apple jelly. They sell these three items at several local grocery stores, at craft fairs in the region, and at their own Friendly Farm Pumpkin Festival for 2 weeks in October. Their three primary resources are cooking time in their kitchen, their own labor time, and the apples. They have a total of 500 cooking hours available, and it requires 3.5 hours to cook a 10-gallon batch of apple butter, 5.2 hours to cook 10 gallons of applesauce, and 2.8 hours to cook 10 gallons of jelly. A 10-gallon batch of apple butter requires 1.2 hours of labor, a batch of sauce takes 0.8 hour, and a batch of jelly requires 1.5 hours. The Friendly family has 240 hours of labor available during the fall. They produce about 6,500 apples each fall. A batch of apple butter requires 40 apples, a 10-gallon batch of applesauce requires 55 apples, and a batch of jelly requires 20 apples. After the products are canned, a batch of apple butter will generate $190 in sales revenue, a batch of applesauce will generate a sales revenue of $170, and a batch of jelly will generate sales revenue of $155. The Friendlys want to know how many batches of apple butter, applesauce, and apple jelly to produce in order to maximize their revenues.

   a. Formulate a linear programming model for this problem.
   b. Solve the model by using the computer

3. If the Friendlys in Problem 1 were to use leftover apples to feed livestock, at an estimated cost savings worth $0.08 per apple in revenue, how would this affect the model and solution?